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Distributions of nonsupersymmetric flux vacua

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Abstract: We continue the study of the distribution of nonsupersymmetric flux vacua in IIb string theory compactified on Calabi-Yau manifolds, as in hep-th/0404116.
We show that the basic structure of this problem is that of finding eigenvectors of the matrix of second derivatives of the superpotential, and that many features of the results are determined by features of the generic ensemble of such matrices, the CI ensemble of Altland and Zirnbauer originating in mesoscopic physics.
We study some simple examples in detail, exhibiting various factors which can favor low or high scale supersymmetry breaking.

Keywords: Superstring Vacua, Supersymmetry Breaking, Statistical Methods.

Contents

1. Introduction

In this work we continue the study of the statistics of flux vacua initiated in [1, 2, 3], and continue the study of supersymmetry breaking vacua begun in [4]. For a general introduction to this problem and its applications, and further references, see [5]. Some more recent works on vacuum statistics and related topics include [6]–[10].

Our main result will be explicit formulae for the density of supersymmetry breaking vacua in an ensemble of effective supergravity field theories (EFT's) in which the superpotential is a "random variable" in a sense we define below. This type of ensemble includes the set of EFT's obtained by compactifying string/M theory with fluxes, in the limit that flux quantization can be neglected. We will give specific results for type IIb flux compactification on Calabi-Yau [11], which at present is the best studied (and most calculable) example, but stress that the techniques (and possibly many features of the results) are more general.

Although we foresee many applications of such results to string duality and phenomenology, a question of primary current interest is the likelihood to discover supersymmetry at the energy range 1 TeV to be probed at LHC, perhaps as the traditional MSSM scenarios, "split supersymmetry" [12, 13], or perhaps other scenarios. As discussed in [1, 5], a systematic way to study this question is to try to count or estimate the number of string vacua which realize the various scenarios. Even with our limited present understanding, it is conceivable that we can argue that some scenarios are so unlikely that we should regard their observation as evidence against string theory, in other words we would have derived a strong prediction from string theory. Admittedly, this possibility depends on optimistic assumptions about the total number of string theory vacua, but in any case we believe vacuum counting will give essential information for any approach to string phenomenology.

The main reason to reconsider traditional attitudes towards naturalness and supersymmetry is that, as becomes clear from a quantitative approach, the advantage gained by low energy supersymmetry is just not that large, compared to the current estimates of the number of string theory vacua; $N_{\rm vac} \gg 10^{100}$ vacua, based both on estimates of the number of flux vacua [14, 2] and on the idea that this large number of vacua is actually needed to solve the cosmological constant problem [15, 16, 14].

To quantify the study of supersymmetry breaking, we adopt the following simplified description of the problem. We need to find the joint distribution of the Higgs mass, supersymmetry breaking scale and cosmological constant,

$$d\mu[M_H, M_{\rm susy}, \Lambda]$$

among "otherwise phenomenologically acceptable" vacua, where

$$M_{\rm susy}^4 = \sum_A |F_A|^2 + \sum_\alpha D_\alpha^2 \,. \tag{1.1}$$

We then evaluate this at the (presumed) values $M_H \sim 100\,{\rm GeV} \sim 10^{-17} M_{\rm planck}$ and $\Lambda \sim 10^{-122} M_{\rm planck}^4$, and look at the resulting distribution of supersymmetry breaking scales. Rough arguments [17, 18, 19] summarized in [5] suggest that for $M_H^2 \geq M_{\rm susy} M_{\rm planck}$, this goes as

$$d\mu[M_H, M_{\rm susy}, \Lambda] \sim \frac{dM_H^2}{M_{\rm susy}^4/M_{\rm planck}^2} \frac{d\Lambda}{M_{\rm planck}^4} d\mu[M_{\rm susy}] \,. \tag{1.2}$$

To get a sense of what this means, consider a simple power law ansatz,

$$d\mu[M_{\rm susy}] \sim M_{\rm susy}^\alpha dM_{\rm susy}^2 \,.$$

Given such a distribution, the distribution of physical vacua would be weighed towards high scales if $\alpha > 2$, showing that it would not take a huge growth in the number of vacua with breaking scale to dominate the advantage in solving the hierarchy problem. [18, 17] Another way to make this point is to note that the ratio $(M_{\rm susy\ high}/M_{\rm susy\ low})^4$ for two plausible guesses at the supersymmetry breaking scale, the high scale $M_{\rm susy\ high} \sim 10^{16}$ GeV and the intermediate scale $M_{\rm susy\ low} \sim 10^{10}$ GeV, is 10^{24}. This is not a large number in the present context.

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Of course the true distribution of supersymmetry breaking scales in string/M theory vacua will be far more complicated than this simple power law ansatz, with various components reflecting various supersymmetry breaking mechanisms and other structure in the problem. While it will clearly take a great deal of work to form any real understanding of this true distribution, we will see in the following that certain generic features of the problem, following just from the the structure of supergravity and of generic distributions of EFT's, do translate into specific properties and factors in the distributions, which might well be shared by the true distribution. Furthermore, many aspects of string compactification which at first sight look highly significant, such as mechanisms to produce hierarchically small scales, can turn out to be no more important than these generic properties in the final statistics.

We turn to detailed considerations of the problem of counting pure F breaking vacua in sections 2 and 3. Starting from a precise ensemble of EFT's such as those obtained by flux compactification, this is a mathematically well defined problem, but not a simple one. While we will obtain precise formulae as in [2, 4], to some extent a more useful description of the basic result is that the distribution of supersymmetry breaking vacua is similar to that of supersymmetric vacua, with certain "correction factors" which we will explain in detail. We also discuss the likely effects of taking flux quantization into account.

In section 4 we make some comments on the distribution obtained by supersymmetry breaking by antibranes, or more generally by D terms.

Section 5 contains conclusions.

2. Preliminary considerations

Our starting point is the $\mathcal{N}=1$ supergravity potential and its derivatives,[1]

$$\begin{aligned} V &= e^{\mathcal{K}}\left(g^{a\bar{b}}D_aW\bar{D}_{\bar{b}}\bar{W}-3|W|^2\right)+D^2 && (2.1)\\ \partial_aV &= e^{\mathcal{K}}(D_aD_bW\bar{D}^b\bar{W}-2D_aW\bar{W}) && (2.2)\\ D_a\partial_bV &= e^{\mathcal{K}}(D_aD_bD_cW\bar{D}^c\bar{W}-D_aD_bW\bar{W}) && (2.3)\\ \bar{D}_{\bar{a}}\partial_bV &= e^{\mathcal{K}}(R^d{}_{c\bar{a}b}D_dW\bar{D}^c\bar{W}+g_{b\bar{a}}D_cW\bar{D}^c\bar{W}-D_bWD_{\bar{a}}\bar{W}-\\ &\quad -2g_{b\bar{a}}W\bar{W}+D_bD_cW\bar{D}_{\bar{a}}\bar{D}^c\bar{W})\,. && (2.4)\end{aligned}$$

For simplicity we will first assume the D terms are zero or at least independent of the fields (until section 4).

The approach we will take to finding statistics of vacua is to summarize the sum over all choices which go into the potential eq. (2.1), in terms of a joint distribution $d\mu$ of the superpotential W and its derivatives, evaluated at a point z in the space of chiral field vevs,

[1]The covariant derivatives D_a are both Kähler and metric covariant, i.e. when acting on W and its derivatives include the Kähler connection $\partial_a\mathcal{K}$, and when acting on tensors include the Levi-Civita connection Γ^b_{ac}. R is the curvature of the cotangent bundle, i.e. $R^d{}_{ca\bar{b}}\,X_d \equiv [\nabla_a,\bar{\nabla}_{\bar{b}}]X_c = \bar{\partial}_{\bar{b}}(g^{\bar{e}d}\partial_a g_{c\bar{e}})\,X_d$. At a vacuum $dV=0$, it does not matter whether the outer derivatives are covariant derivatives or ordinary partial derivatives, because $DdV=d^2V$. We set $M_p=1$.

a candidate vacuum. To evaluate all of eqs. (2.2)–(2.4) at a point z, the joint distribution must describe $W(z)$, $\mathcal{K}(z, \bar{z})$, and up to three derivatives of W at z.

The explicit dependence of these quantities on $\mathcal{K}(z)$, the Kähler potential at z, can be removed by either making redefinitions such as $\hat{W} = e^{\mathcal{K}/2}W$ (as in [4]), or equivalently making a Kähler-Weyl transformation to set $\mathcal{K}(z, \bar{z}) = 0$. This does not change any of the equations because $e^{\mathcal{K}}$ is covariantly constant with respect to D_a. Let us now assume this has been done, and use the notations

$$F_A = D_A W(z)\,; \qquad Z_{AB} = D_A D_B W(z)\,; \qquad U_{ABC} = D_A D_B D_C W(z) \tag{2.5}$$

where capital indices A, B etc. are defined to be orthonormal complex indices with respect to the Kähler metric at z. We retain the symbol W for $W(z)$. The tensors Z and U are symmetric in all indices.

We can then rewrite eqs. (2.2)–(2.4) as

$$\partial_A V = Z_{AB}\bar{F}^B - 2F_A\bar{W} \tag{2.6}$$
$$D_A\partial_B V = U_{ABC}\bar{F}^K - Z_{AB}\bar{W} \tag{2.7}$$
$$\bar{D}_A\partial_B V = R_{ABCD}F^C\bar{F}^D + \delta_{AB}|F|^2 - \bar{F}_A F_B - 2\delta_{AB}|W|^2 + \bar{Z}_{AC}Z_{BD}\,. \tag{2.8}$$

Given a joint distribution

$$d\mu[W, F_A, Z_{AB}, U_{ABC}]\,, \tag{2.9}$$

such as might come from summing over all choices of flux in a given compactification, we will then compute densities such as

$$\rho(z) = \int d\mu\ \delta(V'(z))|\det V''(z)|\theta(V'')$$

in terms of the joint distribution.

2.1 Distributions

The joint distribution eq. (2.9) for an integral over fluxes in IIb on CY in the large volume limit was worked out in [4]. To summarize, in these effective theories the Kähler potential $\mathcal{K}$ is independent of the flux, while the GTVW formula

$$W = \int G \wedge \Omega(z) \tag{2.10}$$

tells us that the superpotential is linear in the flux G. Thus, $W(z)$ and all of its derivatives are linear in the flux, and one can change variables in the integral over fluxes to find a simple joint distribution in which W, F_0, F_I and Z_{0I} are independent. Here the index '0' refers to the dilaton, and $I = 1, \ldots, n$ to the complex structure moduli. Using threefold special geometry, for the particular case of IIb flux vacua, the variables Z_{AB} and U_{ABC} are determined in terms of $Z_I \equiv Z_{0I}$ and F_A; for example

$$Z_{IJ} = \mathcal{F}_{IJK}\bar{Z}^K\,.$$

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Thus, the resulting distribution is

$$d\mu_{IIb} = d^2W\, d^{2n+2}F_A\, d^{2n}Z_{0I}\, \delta(L - |W|^2 + |F|^2 - |Z_{0I}|^2) \tag{2.11}$$

with the other variables determined in terms of these. One could also express the distribution for the original problem with quantized fluxes in this way, as a sum over lattice points embedded in the parameter space in a way determined by z and the periods.

The superpotential eq. (2.10) does not include Kähler moduli and as in [4] we will ignore these moduli. Of course, in an exact description, these must be taken into account. In particular, their stabilization requires a sufficient number of nonperturbative contributions to the superpotential [20, 21, 22], which may or may not be present in a given model. Moreover, to stabilize the volume at a reasonably large value, moderately small values of W_{flux} are needed. However, if thus stabilized, the Kähler sector has relatively little influence on the statistics and properties of vacua, because its contribution to the potential is exponentially small, and because by far the main degeneracy of vacua comes from the different choices of flux. In particular, inclusion of Kähler moduli typically only produces small shifts in the vacua found by ignoring them. This was made more precise in [21] section 4.1, and [7]. The constraint of small W_{flux}, needed to stabilize the model in a controlled (large radius) regime, merely reduces the number of suitable vacua by a factor $|W_{\text{flux}}|^2$ [4], and anyway this condition will be met automatically for the phenomenologically most relevant vacua we will study in the following, namely those with $\Lambda \sim 0$ and $M_{\text{susy}}^2 \ll 1$.

The requirement of a sufficient number of nonperturbative contributions is more subtle, but the results of [21, 22] suggest that many models satisfying this should exist, although completely explicit constructions tend to be computationally complex. We suspect that major progress on this point would come from developing other ways of understanding Kähler moduli stabilization. For example, mirror symmetry suggests the existence of type-IIB non-Calabi-Yau deformations dual to turning on IIA NS-NS flux, which should thus be described by a similar superpotential depending on the IIB Kähler moduli. In this case, the Kähler sector would also contribute significantly to the counting of vacua, but since it would be governed by a flux type superpotential, it would also fit into the general class of supergravity ensembles for which the analysis below should be valid. We discuss this point a bit further in section 3.2.2.

In any case, this discussion, together with the fact that IIB flux vacua are computationally very accessible due to the underlying special geometry structure, justifies the claim that such ensembles are good models for the statistics of string vacua.

One could do a similar computation for any class of models in which $\mathcal{K}$ and W are computable. Generically, one would expect to obtain a similar distribution, in which a sum over K flux (or other) parameters leads to a rough independence for the first K quantities in the distribution. In theories (such as the heterotic string) with fewer fluxes than the IIb string, this would lead to constraints between the Z's and the (W, F) variables, while in theories with more parameters (such as F theory on fourfolds) one has more parameters and thus might expect a more generic ensemble of matrices Z_{IJ}.

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While it remains to be seen which distribution best represents the full set of string/M theory vacua, it seems plausible that this would be the one with the most free parameters, i.e. the fourfolds. One furthermore expects quantum corrections to these classical flux superpotentials. Thus another interesting ensemble, which might usefully represent this more generic distribution, is simply to take all of the parameters to be independent,

$$d\mu_G = d^2W d^{2m} F_A d^{m(m+1)} Z_{AB} \delta(L - |W|^2 + |F|^2 - |Z|^2) \tag{2.12}$$

where the delta function is an ansatz for a cutoff analogous to the one which made the number of IIb flux vacua finite. This will turn out to be related to distributions previously considered in random matrix theory, as we discuss in section 2.4.

To complete the specification of such a model ensemble, one must define the variables U_{ABC}. Now in the fourfolds and other examples we might model this on, the U variables would still be determined in terms of the others, which might be important. One might also argue that a better model ensemble would be a simple proposal which keeps more of the structure of Calabi-Yau moduli spaces and their degenerations. These are interesting ideas for future work, but as we will argue below eq. (2.12) already shares interesting features with more realistic distributions.

In any case, we will keep most of the discussion general, and not assume any *a priori* relation between the variables in eq. (2.9).

2.2 Solving the equations $V' = 0$

The equations $dV = 0$ are quadratic in the parameters, and the best way to exhibit their structure is to rewrite them as a matrix $N(W, Z)$ depending on $(W, \bar{W}, Z, \bar{Z})$ acting on the vector F_A, as

$$0 = \begin{pmatrix} \partial_A V \\ \bar{\partial}_A V \end{pmatrix} = \begin{pmatrix} -2\bar{W} & Z_{AB} \\ \bar{Z}_{AB} & -2W \end{pmatrix} \begin{pmatrix} F^B \\ \bar{F}^B \end{pmatrix} \tag{2.13}$$

$$= \begin{pmatrix} -2|W| & e^{-i\theta} Z_{AB} \\ e^{i\theta} \bar{Z}_{AB} & -2|W| \end{pmatrix} \begin{pmatrix} e^{-i\theta} F^B \\ e^{i\theta} \bar{F}^B \end{pmatrix} \tag{2.14}$$

where $e^{i\theta} = W/|W|$.

Define the matrix $N(W, Z)$ to be the second of these matrices (i.e. the matrix depending on $|W|$), and write

$$N = M - 2|W| \cdot \mathbf{1}$$

in terms of the matrix

$$M = \begin{pmatrix} 0 & e^{-i\theta} Z_{AB} \\ e^{i\theta} \bar{Z}_{AB} & 0 \end{pmatrix}. \tag{2.15}$$

Since Z_{AB} is symmetric, the matrix M is hermitian, so it has an orthonormal basis of eigenvectors. Thus, we see that non-supersymmetric vacua (solutions of $V' = 0$ with $F \neq 0$) correspond to eigenvectors of the matrix M with eigenvalue $2|W|$.

Clearly the matrix M is not a generic hermitian matrix; it has additional symmetry properties. First, its eigenvectors come in pairs with opposite eigenvalues $(+\lambda_a, -\lambda_a)$. These eigenvalues are independent of θ; this follows because their squares are the eigenvalues of the hermitian matrix $Z\bar{Z}$. The corresponding eigenvectors are

$$\Psi_a^+ = \begin{pmatrix} e^{-i\theta/2}\psi_a \\ e^{i\theta/2}\bar{\psi}_a \end{pmatrix}, \qquad \Psi_a^- = \begin{pmatrix} ie^{-i\theta/2}\psi_a \\ -ie^{i\theta/2}\bar{\psi}_a \end{pmatrix}, \tag{2.16}$$

where ψ_a solves

$$Z\bar{\psi}_a = \lambda_a \psi_a \tag{2.17}$$

and we take $\lambda_a \geq 0$. Because the ψ_a are eigenvectors of the hermitian matrix $Z\bar{Z}$, we can take them to be orthonormal. Then, defining the unitary matrix U by $U_{Ab} \equiv \bar{\psi}_{b,A}$, we have

$$Z = U\lambda U^T \tag{2.18}$$

where $\lambda = \mathrm{diag}(\lambda_a)$. Any symmetric complex matrix can be decomposed in this way.

Thus, the generic supersymmetry-breaking solutions to (2.13) can be written as

$$2W = \lambda_a e^{i\theta}\,; \qquad F = f e^{i\theta/2}\psi_a \tag{2.19}$$

with $f \in \mathbb{R}$. Varying θ in $[0, 2\pi]$, this fills out a one complex dimensional subspace.

2.3 Masses of moduli

We have just seen that non-supersymmetric solutions of $V' = 0$ correspond to eigenvectors of a matrix M with eigenvalue $2|W|$. Thus, in ensembles such as the IIb flux ensemble in which the parameters W and F_A can be freely varied, such vacua are **generic**.

Of course, the vacua of most interest are metastable, i.e. the bosonic mass matrix V'' has no negative eigenvalues. This mass matrix is

$$d^2V = (M + |W|)(M - 2|W|) + V_1'' + V_2'' \tag{2.20}$$

where the successive terms are of order F^0, F^1 and F^2.

The F^0 term is the product of the matrix N above with the matrix

$$H = M + |W| \cdot \mathbf{1}$$

which is the same as the matrix $d^2|W|$ whose determinant enters into the density of supersymmetric vacua [4]. The higher order terms are

$$V_1'' = \begin{pmatrix} 0 & S_1 \\ \bar{S}_1 & 0 \end{pmatrix}, \qquad S_1 = U_{ABC}\bar{F}^C$$

and

$$V_2'' = \begin{pmatrix} S_2 & 0 \\ 0 & \bar{S}_2 \end{pmatrix}, \qquad S_2 = R_{A\bar{B}C\bar{D}}\bar{F}^C F^D + \delta_{AB}|F|^2 - F_A\bar{F}_B\,.$$

We now assume $|F| \ll M_p \equiv 1$. In this case, the eigenvectors $\Psi_a^\pm$ of M defined in (2.16), are approximate eigenvectors of d^2V. Let us take the eigenvalues λ_A to be ordered,

$$0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m \, . \tag{2.21}$$

The values of V'' in the Ψ_a directions are

$$\begin{aligned}(m_a^\pm)^2 &= \langle \Psi_a^\pm, d^2V\Psi_a^\pm \rangle \\ &= (\pm\lambda_a + |W|)(\pm\lambda_a - 2|W|) \pm 2\,\mathrm{Re}(e^{i\theta}\bar{\psi}_a S_1 \bar{\psi}_a) + 2\,\bar{\psi}_a S_2 \psi_a \, .\end{aligned} \tag{2.22}$$

By (2.13), one of these eigenvectors, call it Ψ_F^+, is proportional to $(e^{-i\theta}F^B\, e^{i\theta}\bar{F}^B)$, with $\lambda = 2|W|$. There will also be a complementary eigenvector Ψ_F^- with eigenvalue $-2|W|$.

Let us first consider variations of the moduli proportional to the other eigenvectors. Suppose the corresponding eigenvalue λ_a is greater than $2|W|$. In this case, the first term in (2.22) will be positive, and if the gap $\lambda_a - 2|W|$ is sufficiently large (as will be the case for generic Z when F is small), this term dominates the higher order terms. On the other hand, positive eigenvalues less than $2|W|$ will produce a negative m^2, unless they are very close to $2|W|$ and the higher order terms are fine tuned to compensate for the negative lowest order term. Thus, the bulk of tachyon-free nonsupersymmetric vacua with sufficiently small F will come from solutions for which $2|W|$ equals the **lowest** eigenvalue λ_1.

We now consider $(m_1^\pm)^2 = (m_F^\pm)^2$. For m_1^+, the first term in eq. (2.22) vanishes, so V'' in this direction is given by the matrix element of the higher order terms:

$$(m_F^+)^2 = \frac{2}{|F|^2}\Big(\mathrm{Re}(e^{2i\theta}U_{ABC}\bar{F}^A\bar{F}^B\bar{F}^C) + R_{A\bar{B}C\bar{D}}\bar{F}^A F^B \bar{F}^C F^D\Big) \, . \tag{2.23}$$

Similarly, the matrix element for the complementary eigenvector with eigenvalue $-2|W|$ is

$$(m_F^-)^2 = 4|W|^2 + \frac{2}{|F|^2}\Big(-\,\mathrm{Re}(e^{2i\theta}U_{ABC}\bar{F}^A\bar{F}^B\bar{F}^C) + R_{A\bar{B}C\bar{D}}\bar{F}^A F^B \bar{F}^C F^D\Big) \, . \tag{2.24}$$

There are now two cases to distinguish. The simplest case is $|F| \ll |W|$, but this can only be compatible with $V = |F|^2 - 3|W|^2 + |D|^2 \sim 0$ if $|D| \sim |W|$, i.e. the supersymmetry breaking is dominated by the D terms. If we have $D = 0$ or even $|D| \sim |F|$, then $|F| \sim |W|$ and we must take this into account.

In the first case, and for generic U_{ABC}, the first term in eq. (2.23) will dominate, so $(m_F^+)^2$ will be positive for half the integration domain of θ. Furthermore, if $|W|$ is not too small (greater than $O(\sqrt{F})$), the first term in eq. (2.24) will dominate and $(m_F^-)^2$ will also be positive. Hence the requirement of metastability puts only a mild constraint on the integration domain in this case.

On the other hand, if $|F| \sim |W|$, the first term in eq. (2.24) is of order F^2 and no longer dominates. Instead, for generic U and θ, either $(m_F^+)^2 < 0$ or $(m_F^-)^2 < 0$, as the $O(F)$ term appears with different signs in these. Thus, to get a metastable vacuum with zero (or positive) cosmological constant, we have to fine-tune the $O(F)$ term such that it becomes smaller than the $O(F^2)$ term:

$$\mathrm{Re}(e^{i\theta/2}U_{ABC}\bar{\psi}^A\bar{\psi}^B\bar{\psi}^C) < O(F) \, . \tag{2.25}$$

This can be achieved by tuning θ in an interval of size $\sim |F|$, which, if the measure for θ is otherwise uniform, will give an additional suppression of the expected number of vacua by a factor of $|F|$.

However, because the diagonal mass matrix elements are now tuned one order in F smaller, we have to be a bit more careful and also consider the off diagonal matrix element between Ψ_1^+ and Ψ_1^-.[2] This is

$$\langle \Psi_1^-, d^2V\,\Psi_1^+\rangle = 2\,\mathrm{Im}(e^{i\theta}\bar\psi_1 S_1 \bar\psi_1) = \frac{2}{|F|^2}\,\mathrm{Im}(e^{2i\theta}U_{ABC}\bar F^A \bar F^B \bar F^C)\,. \tag{2.26}$$

Denoting $s_1 \equiv e^{i\theta}\bar\psi_1 S_1 \bar\psi_1$, $s_2 \equiv \bar\psi_1 S_2 \psi_1$, the determinant of the 2×2 mass matrix in the 1-directions is thus

$$\det = 4\left(|W|^2(\mathrm{Re}\,s_1 + s_2) - |s_1|^2 + s_2^2\right). \tag{2.27}$$

Note that if $W \sim F$, the leading term in the expansion in powers of F is generically $-|s_1|^2 \sim F^2$, and this is always negative *independent* of θ. Therefore, to avoid a tachyon, we really need $|s_1| < O(F)^2$, so (2.25) gets replaced by

$$|U_{ABC}\bar\psi_1^A \bar\psi_1^B \bar\psi_1^C| < O(F)\,. \tag{2.28}$$

Assuming approximate uniform distribution of this complex component of U, this metastability constraint will therefore give an additional $O(F^2)$ suppression rather than the $O(F)$ we deduced neglecting the off-diagonal element. The masses $m_1^\pm$ on the other hand will still be of order F.

In any case, the overall conclusion of this analysis is that metastability is a relatively mild constraint, in the sense that it does not drastically reduce the number of nonsupersymmetric vacua. In general, we find that a rough fraction $1/n$ of all nonsupersymmetric critical points are metastable vacua (since we can only use one of the n eigenvalues). This is much larger than the naive estimate of 2^{-2n}, obtained by taking the m_i^2 independent and symmetrically distributed around zero. This is consistent with the intuition that, for $|F| \ll |W| \ll 1$, the situation is similar to that for global supersymmetry, in which stability is automatic. But the arguments here apply far more generally than in this limit.

Nevertheless, metastability can significantly influence the distribution: the distribution of the lowest eigenvalue can be different from that of an arbitrary eigenvalue, and constraints such as eq. (2.28) are important.

2.4 Degeneracies and the dimension of the solution space

We have just seen that for generic parameters W and Z, the space of possible supersymmetry breaking parameters F is one complex dimensional, the eigenvector of M with the lowest positive eigenvalue, multiplied by a general phase factor.

However, at special loci in Z space, the matrix M might have k-fold degenerate eigenvalues. In this case, the vector F will vary in a k real dimensional space, and varying θ will fill out a $k+1$ real dimensional subspace. This is the structure which might lead to power law growth of the number of vacua with the supersymmetry breaking scale, as suggested in [18, 17].

[2] We thank Michael Dine, Deva O'Neil and Zheng Sun for pointing this out to us.

An explicit example in which this is realized is the "anti-supersymmetric branch" of IIb flux vacua, with $Z = W = 0$ or equivalently imaginary anti-self-dual flux. Now these vacua are physically not interesting, because they have cosmological constant at least of the order of the string scale [4], and moreover it can be shown that they always have a tachyonic mode. But the number of these vacua does grow as a high power of F, so we should not immediately dismiss the idea that some physically sensible subset of the vacua behaves in the same way.

However, if we ask what we need to get degenerate eigenvalues in other circumstances, we find that they are non-generic, in the sense that one must tune more than $k-1$ parameters to get a k-fold eigenvalue degeneracy. Because of this, it will turn out that these "higher branches" of non-supersymmetric vacua are of lower dimension than the primary branch, and thus will not contribute to the overall volume estimate, and thus to the leading large L asymptotics for the number of vacua.

The arguments are analogous to the more familiar discussion of degenerate eigenvalues in families of hermitian matrices, so let us review this first. For hermitian matrices, one must tune k^2-1 real parameters to get a k-fold degeneracy. The simplest way to see this is to consider the change of variables to eigenvalues and eigenvectors $M = U^\dagger \lambda U$. This is generically unambiguous up to permutation of eigenvalues, and up to a $\mathrm{U}(1)^n$ left action on U. Thus the n^2 real parameters of M go over to n eigenvalues and n^2-n coordinates on $\mathrm{U}(n)/\mathrm{U}(1)^n$. However, a diagonal matrix with a k-fold degenerate eigenvalue is preserved by conjugation by an element of $\mathrm{U}(k)$, and thus such matrices form a stratum in the larger space of all matrices of real codimension $k-1$ (required to tune the eigenvalues) plus k^2-k (the dimension of the stabilizer group).

To make the analogous argument for the case at hand, we consider the change of variables eq. (2.18). Now the $n(n+1)$ real parameters in Z are rewritten as n eigenvalues and n^2 parameters of the $\mathrm{U}(n)$ group element V. On the other hand, a k-fold degenerate diagonal matrix will be preserved by a unitary V satisfying $VV^T = 1$, in other words an element of $\mathrm{SO}(k)$. Thus the degenerate matrices form a stratum of total real codimension $(k+2)(k-1)/2$, the dimension $k(k-1)/2$ of $\mathrm{SO}(k)$ plus $k-1$ for the eigenvalues.

A related way to see this is to consider the change of variables from the Lebesgue measure on matrix elements, to eigenvalues and unitary group elements. For the hermitian matrix, this is

$$d^{N^2}M = [dU] \prod_i d\lambda_i \prod_{i<j} (\lambda_i - \lambda_j)^2$$

where $[dU]$ is the invariant measure on $\mathrm{U}(n)$. This exhibits the extra real codimension $k(k-1)$ as an explicit scaling dimension in the jacobian, in analogy to the scaling $r^{D-1}dr$ of Lebesgue measure in polar coordinates.

The analogous matrix ensemble for the present case is determined by the symmetry properties of eq. (2.15). Introducing a triplet of Pauli matrices σ_i to act on the 2×2 block structure of eq. (2.15), these are

$$M = M^\dagger = -\sigma_3 M \sigma_3 = -\sigma_2 M^* \sigma_2 \,. \tag{2.29}$$

These symmetry properties are formally equivalent to those of the "CI ensemble" of Altland and Zirnbauer [23]. This was introduced as an ensemble of hamiltonians of mesoscopic systems, and in that context these symmetry properties are time reversal invariance and spin rotation invariance. While the physics and the interpretation of the symmetry properties is different here, and the actual ensembles arising from string theory (as discussed in section 2.1) are a subset of the CI ensemble, certain properties of the CI ensemble and specifically the structure of eigenvalue degeneracies should be shared by the stringy ensembles.

Thus, we consider the CI ensemble measure, in which the different matrix elements (consistent with eq. (2.29)) are independently distributed; this is just the ensemble eq. (2.12). We then perform the change of variables

$$M = U^\dagger \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} U\,.$$

The CI measure transforms to

$$d\mu[\lambda, U] = [dU] \prod_a d(\lambda_a^2) \prod_{a<b} |\lambda_a^2 - \lambda_b^2| \tag{2.30}$$

and we again exhibit the codimension $k(k-1)/2 + (k-1)$ of the symmetry locus as a scaling dimension, as well as the fact that the true invariants of M are the squares of the eigenvalues.

Now, given a set of k eigenvectors ψ_a, $1 \le a \le k$, all with eigenvalue λ, the analog of eq. (2.19) is

$$2W = \lambda e^{i\theta}\,; \qquad F = f^a e^{i\theta/2}\psi_a \tag{2.31}$$

where f^a is a real k component vector. Thus, while the associated locus of nonsupersymmetric vacua is $k+1$ real dimensional, and the integral over fluxes $d^{2n}F$ would lead to an additional power law growth $|F|^{k-1}$ (compared to the case $k=1$), the variables Z_{IJ} must satisfy $(k+2)(k-1)/2$ additional real constraints, leading to a total real codimension $(k+2)(k-1)/2 - (k-1) = k(k-1)/2$ for the k-fold degenerate branch. Thus already for $k=2$ these branches are of lower dimension in the generic case, and will have volume zero.

A similar discussion applies in the special case of a k-fold degenerate zero eigenvalue, but now both Ψ^+ and Ψ^- can be superposed in eq. (2.19), so the resulting locus of nonsupersymmetric vacua is $2k$ real dimensional. On the other hand, a k-fold degeneracy at zero generically appears in real codimension $2(k-1) + k(k-1) = (k+2)(k-1)$, so again the resulting branch is of lower dimension.

There is a loophole in these arguments, which is the assumption of genericity. If the matrix Z_{IJ} depends on the fluxes and fields in a non-generic way, such that a k-fold degeneracy appears with real codimension $k-1$, then the corresponding branch of vacua would have the same dimension as the primary branch. This is what happens in the "anti-supersymmetric" branch: because of the relation $Z_{IJ} = \mathcal{F}_{IJK}\bar{Z}^K$, one need tune only the n complex parameters Z_{0I} to obtain $Z=0$ and a $2n$-fold degeneracy at zero, so this branch is again of the full dimension. Since the matrices Z_{IJ} obtained in IIb flux compactification are definitely not generic, we need to understand this point.

A familiar way to get a k-fold eigenvalue degeneracy in codimension $k-1$ for a family of hermitian matrices is to take the sum of k **commuting** matrices,

$$M = \sum t_i M_i \qquad \text{with } [M_i, M_j] = 0\,. \tag{2.32}$$

In this case, since can simultaneously diagonalize the M_i, the problem reduces to considering families of eigenvalues, for which there is no difficulty in tuning to degeneracies. Conversely, if the family is a linear sum eq. (2.32), this is the only way to get eigenvalue degeneracies; if some $[M_i, M_j] \neq 0$ then eigenvalue repulsion always makes the codimension of a k-fold degeneracy (reachable by tuning parameters which include t_i and t_j) higher than $k-1$.[3]

The situation for matrices in the symmetry class CI is similar, and governed by the same criterion eq. (2.32) applied to the matrix eq. (2.15). This implies that a k-fold eigenvalue degeneracy in codimension $k-1$. is possible only if Z is a linear sum of k matrices $\hat{Z}_i$ which commute after applying a similarity transformation; in other words if there exists a unitary U such that

$$[U\hat{Z}_i U^T, U\hat{Z}_j U^T] = 0\,. \tag{2.33}$$

Thus generic sums of n matrices will exhibit the generic behavior of an n-parameter family of matrices, in other words eigenvalue repulsion and no $n+1$-fold degenerate eigenvalues.

The most obvious way to get the structure eq. (2.32) and independent F breaking parameters, is the case that the effective supergravity lagrangian can be written as the sum of two (or more) completely independent lagrangians, each depending on a distinct subset of the fields. This fits with the intuition that in a model with several decoupled hidden sectors, supersymmetry breaking in the various sectors would be independent, but only if the different sectors satisfy a very strong decoupling requirement. One way to get this is to consider limits in moduli space, and following up this idea leads to the question, what would be the consequences of a nearly degenerate eigenvalue. From eq. (2.20), this would seem to lead to a nearly massless boson, but no obvious enhancement of the number of vacua. This case deserves closer study, but we leave this for future work.

It turns out that less simple but naturally occuring examples can also satisfy eq. (2.32) and lead to degenerate eigenvalues; we will see this below for the case $T^6/\mathbb{Z}_2\times\mathbb{Z}_2$. However, this example has an unusual degree of symmetry, and we have seen no sign of the structure eq. (2.32) in other Calabi-Yau compactifications.

2.5 Some physical comments

While we suspect that some of these observations have been made before, our discussion does not look much like the existing discussions of supersymmetry breaking in the literature, which made it hard for us to compare with previous work. There are two reasons for

[3] This can be seen by considering the free classical mechanics with lagrangian $\mathrm{tr}(dM/dt)^2$, and the solution $M = (1-t)M_1 + tM_2$. Changing variables to eigenvalues and unitaries, if $[M_1, M_2] \neq 0$ there will be a non-zero angular momentum $L = [M, \dot{M}]$, and a potential $L_{ij}^2/(\lambda_i - \lambda_j)^2$ repelling the trajectory from the degeneracy.

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this. First, in model building terms, the considerations here would describe the origin of supersymmetry breaking in hidden sectors, and not its mediation or effects on the observable sector. Second, we have tried hard to use the underlying geometry to simplify the problem to its essentials, perhaps at the cost of some physical intuition. Let us make a few comments to remedy this.

The matrix M is closely related to the supergravity fermion mass matrix, which can be read off from the lagrangian ([24], (23.3)):

$$\mathcal{L}_F = W\bar{\psi}_a\sigma^{ab}\bar{\psi}_b + \bar{W}\psi_a\sigma^{ab}\psi_b + \frac{i}{\sqrt{2}}F_A\chi^A\sigma^a\bar{\psi}_a + \frac{i}{\sqrt{2}}\bar{F}_A\bar{\chi}^A\sigma^a\psi_a + \frac{1}{2}Z_{AB}\chi^A\chi^B + \frac{1}{2}\bar{Z}_{AB}\bar{\chi}^A\bar{\chi}^B$$

where ψ^a is the gravitino, χ_A are the fermionic partners of the moduli z^Z, and $\bar{\psi}^a$ and $\bar{\chi}_A$ are related to these by complex conjugation. Making the change of variables $\psi^a \to e^{i\theta/2}\psi^a$ and $\chi_A \to e^{-i\theta/2}\chi_A$, and decomposing χ into the eigenvectors eq. (2.17), one finds that the eigenvector ψ_F with eigenvalue $2|W|$ is the goldstino, while the other eigenvectors ψ_a lead to fermions with Majorana masses $m_A = \lambda_A$. The condition that M has an eigenvalue $2|W|$ is also the same as the condition for a scalar in a supersymmetric vacuum to go massless, as discussed in section 3.2 of [4]. This is not a coincidence; it is because different physical branches of the space of vacua attach at second order phase transitions, i.e. points at which a scalar field becomes massless. As we vary N, when its kernel jumps, a modulus will go massless. Note however that there are two types of variation of N, those which come from varying the moduli z^A, but also those which come from varying the effective lagrangian data (W, F, Z). While all supersymmetry breaking branches are connected to the supersymmetric branches by varying N, these will be physical second order transitions only if they are connected by varying moduli.

One way to see the origin of the stability condition $\lambda_a \geq \lambda_1 = 2|W|$ is that the bosonic mass matrix eq. (2.6) has a universal term $\delta_{AB}(|F|^2 - 2|W|^2)$, which for $|F| \ll |W|$ is a universal tachyonic contribution. This not much emphasized in the literature, which usually considers the case $D = 0$ or equivalently $|F|^2 = 3|W|^2$, in which case this contribution is positive. However it is quite important in general, especially in the case $|F| \ll |D| \sim |W|$.

3. Distributions of non-supersymmetric vacua

We proceed to work out the distribution of supersymmetry breaking parameters,

$$d\mu[F] = \int d\mu[W, F, Z, U]\ \delta^{2m}(dV)\,|\det d^2V|\,. \tag{3.1}$$

We start by changing variables from Z_{AB} to its eigenvalues and eigenvectors λ_A and ψ_A. This gives us a joint distribution

$$d\mu[W, F, \lambda_A, \psi_A, U]\,.$$

These variables are redundant under permutations in S_N acting on the indices A. To fix this, we take the λ_A in increasing order, as in eq. (2.21).

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We then solve the constraint $dV = 0$ as in (2.19), in a basis with Z diagonal. Put $\phi_A = \arg(e^{-i\theta/2}F_A)$. We focus on solutions with $2|W| = \lambda_1$, because this is where the bulk of the metastable vacua are located, as argued in section 2.3. Factorizing $\delta(N\cdot F)$ (with $N\cdot F$ as in eq. (2.13)) turns

$$\begin{aligned} d^{2m}F\ \delta^{(2m)}(N\cdot F) &= d^{2m}F\ \prod_A \delta[\mathrm{Re}(\lambda_A\bar{F}_A - 2|W|e^{-i\theta}F_A)]\,\delta[\mathrm{Im}(\lambda_A\bar{F}_A - 2|W|e^{-i\theta}F_A)] \\ &= d^{2m}F\ \prod_A \delta[|F_A|\cos\phi_A(\lambda_A - 2|W|)]\delta[|F_A|\sin\phi_A(\lambda_A+2|W|)] \\ &= d^{2m}F\ \frac{1}{|F_1|}\delta(\lambda_1 - 2|W|)\,\frac{1}{|F_1|(\lambda_1+2|W|)}\delta(\sin\phi_1)\prod_{A>1}\frac{\delta^2(F_A)}{\lambda_A^2 - 4|W|^2} \\ &= \frac{df}{|f|}\frac{\delta(\lambda_1^2 - 4|W|^2)}{|\det' N|} \end{aligned} \tag{3.2}$$

where $\det' N$ is the determinant with the modes $\Psi_1^{\pm}$ excluded, and F is given by (2.19) with $A = 1$.

For small F, the determinant of d^2V can be approximated by

$$\begin{aligned} \det d^2V &\approx \prod_A (m_A^+)^2(m_A^-)^2 \\ &\approx (m_1^+)^2(m_1^-)^2\prod_{A>1}(|W|^2 - \lambda_A^2)(4|W|^2 - \lambda_A^2) \\ &= -\frac{\det H}{3|W|^2}\det{}'N\,(m_1^+)^2(m_1^-)^2\,. \end{aligned} \tag{3.3}$$

The m_A are the masses introduced in eq. (2.22), and we assumed the λ_A for $A > 1$ to be sufficiently above $\lambda_1 = 2|W|$ such that the higher order terms can be neglected in these m_A. Thus we can extract the $\det' N$ from the numerator, to cancel with the one in eq. (3.2). The $1/3|W|^2$ here cancels the factor in $\det H$ coming from the eigenvectors $\Psi_1^{\pm}$, so this expression is non-singular.

The result for the distribution of nonsupersymmetric vacua is $d\mu[W, F, \lambda, \psi, U]\,\rho$, where

$$\rho = \delta^{2m-1}(F)\,\delta(\lambda_1^2 - 4|W|^2)\,\frac{|\det H|}{3|W|^2}\cdot\frac{(m_1^+)^2(m_1^-)^2}{|F|} \tag{3.4}$$

and $\delta^{2m-1}(F)$ is interpreted as above, and $(m_1^{\pm})^2$ is as in eq. (2.23) and eq. (2.24).

Most of the structure of the result is in the factor $|\det H|$. This is the same as the jacobian appearing in the density of supersymmetric vacua, so the result can be summarized as saying that the density of non-supersymmetric vacua is very similar to that for supersymmetric vacua, with a "correction factor" proportional to the product of the squared masses of the two moduli in the supersymmetry breaking direction.

If we were on a locus with k degenerate eigenvalues, we would have to generalize the above steps: eq. (3.2) would contain a product of k delta functions $\delta(\lambda_i - 2|W|)/F$, we would have a larger kernel of N, and would need to replace eq. (3.3) with the determinant of a $k\times k$ submatrix of $V_1'' + V_2''$. While possible in principle, normally this will happen only in codimension larger than k, as discussed earlier, and thus will not contribute to the final integral.

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If we do not impose conditions on the value of the cosmological constant, we have generically (i.e. for $|W|$ not too small) $(m_1^-)^2 \approx 4|W|^2$ and the distribution becomes

$$\rho = \delta^{2m-1}(F)\,\delta(\lambda_1^2 - 4|W|^2)\,\Theta_+(\theta)\,\frac{|\det H|}{3|W|^2} \cdot \frac{4|W|^2|\,\mathrm{Re}(e^{2i\theta}U_{ABC}\bar{F}^A\bar{F}^B\bar{F}^C)|}{|F|^3}\,. \tag{3.5}$$

Here $\Theta_+(\theta)$ restricts θ to values for which $(m_1^+)^2 > 0$ (which is approximately half its integration domain). Thus we see that the small $|F|$ behavior of the distribution is dF, so the number of generic metastable supersymmetry breaking vacua with supersymmetry breaking scale eq. (1.1) less than M_*

$$\mathcal{N}(M_{\rm susy} < M_*) \sim M_*^2\,. \tag{3.6}$$

If on the other hand we restrict to metastable vacua with $V \sim 0$ (and $D = 0$),[4] we have to multiply this distribution by $\delta(|F|^2 - 3|W|^2) = \delta(|F| - \sqrt{3}|W|)/2|F|$, leading to

$$\rho = \delta(|F| - \sqrt{3}|W|)\,\delta^{2m-1}(F)\,\delta(\lambda_1^2 - 4|W|^2)\,\Theta_+(\theta, F)\,\frac{|\det Z|^2}{4|W|^2} \cdot \frac{(m_1^+)^2(m_1^-)^2}{2|F|^2} \tag{3.7}$$

where $\Theta_+(\theta, F)$ restricts θ and F to values such that $(m_1^\pm)^2 > 0$, which as argued in section 2.3 gives at least an additional $O(F)$ suppression. We also used $|\det H| \approx 3|W|^2 \times \prod_{A>1}\lambda_A^2 = \frac{3}{4}|\det Z|^2$.

Let us now grant that the joint distribution we started with looks like

$$d\mu[W, F, Z, U] \sim d^2W\; d^{2n+2}F\; d\mu \ldots\,;$$

in other words W and each component F_A are roughly uniformly distributed in the complex plane, and independent of each other and the Z's. All this is true in the IIb case. We can then use the delta functions to solve for W and $|F|$, to obtain

$$\rho \sim d\mu[\lambda, U]dF\,\Theta_+(\theta, F)\,\frac{|\det Z|^2}{4|W|^2} \cdot \frac{(m_1^+)^2(m_1^-)^2}{2|F|^2}\,. \tag{3.8}$$

As discussed in section 2.3, metastability forces $(m_1^+)^2 \sim (m_1^-)^2 \sim F^2$ and $\Theta_+(\theta, F) \sim F^2$, leading to

$$\rho \sim d\mu[\lambda]F^4 dF\,.$$

Furthermore, in a generic distribution of superpotentials such as eq. (2.30), one expects

$$d\mu[\lambda] \sim \lambda d\lambda \sim F dF\,,$$

leading to

$$\rho \sim F^5 dF\,.$$

Thus, the number of metastable susy breaking vacua with near-zero cosmological constant and susy breaking scale $M_{\rm susy} < M_*$ goes like

$$\mathcal{N}(M_{\rm susy} < M_*, \Lambda \sim 0) \sim M_*^{12}\,. \tag{3.9}$$

Among such vacua, small supersymmetry breaking scales are disfavored. Note also that this result suggests that metastable de Sitter vacua obtained by pure F-term susy breaking are relatively rare.

[4] By $V \sim 0$ we mean $|V| \ll O(F)$, which can still be much larger than the observed cosmological constant. In particular further quantum corrections to Λ remain approximately within this window.

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3.1 Type-IIB flux vacua

For type-IIB flux vacua (ignoring Kähler moduli), we have [4]:

$$Z_{00} = 0\,, \qquad Z_{0I} \equiv Z_I\,, \qquad Z_{IJ} = \mathcal{F}_{IJK}\bar{Z}^K \tag{3.10}$$

and

$$U_{00I} = 0\,, \qquad U_{0IJ} = \mathcal{F}_{IJK}\bar{F}^K\,, \qquad U_{IJK} = D_I\mathcal{F}_{JKL}\bar{Z}^L + \mathcal{F}_{IJK}\bar{F}^0\,. \tag{3.11}$$

The nonzero curvature components are

$$R_{0\bar{0}0\bar{0}} = -2\,, \qquad R_{I\bar{J}K\bar{L}} = \mathcal{F}_{IKM}\bar{\mathcal{F}}^M{}_{JL} - \delta_{IJ}\delta_{KL} - \delta_{IL}\delta_{JK}\,. \tag{3.12}$$

This gives

$$\begin{aligned}(m_1^{\pm})^2 = {} & \frac{2}{|F|^2}\Big(|\mathcal{F}_{IJK}\bar{F}^J\bar{F}^K \pm 2e^{-2i\theta}F_0F_I|^2 - 2|F|^4 \pm \mathrm{Re}(e^{2i\theta}D_I\mathcal{F}_{JKL}\bar{Z}^I\bar{F}^J\bar{F}^K\bar{F}^L)\Big) + \\ & +\delta_{\pm,-}\,4|W|^2\,, \end{aligned} \tag{3.13}$$

so indeed when $F \sim W$ and $D\mathcal{F} \sim O(1)$, to keep both squared masses positive, one needs an order F fine-tuning of the term proportional to $D\mathcal{F}$. At large complex structure, $D\mathcal{F} = 0$ so no such fine tuning is needed.[5] On the other hand, if the number of moduli is large, this constitutes only a tiny part of the moduli space and hence of the number of vacua. Near a conifold degeneration, both $D\mathcal{F}$ and $\mathcal{F}$ blow up, with $D\mathcal{F} \sim \mathcal{F}^2$, so the same kind of fine-tuning is needed as in the generic case.

3.1.1 One complex structure modulus

We turn to some more explicit results, beginning with the simplest case of one complex structure modulus. We will see that this illustrates some but not all features of the generic discussion above.

Letting $Z \equiv Z_{01}$, the matrix $Z_{IJ} = D_ID_JW$ is

$$Z_{IJ} = \begin{pmatrix} 0 & Z \\ Z & \mathcal{F}\bar{Z} \end{pmatrix},$$

and the eigenvalues λ of M satisfy

$$(\lambda^2 - |Z|^2)(\lambda^2 - |Z|^2(1 + |\mathcal{F}|^2)) - |\mathcal{F}|^2|Z|^4 = 0\,.$$

More explicitly, $\lambda_{1,2} = \frac{1}{2}(|\mathcal{F}| \pm \sqrt{|\mathcal{F}|^2 + 4})|Z|$. Near a conifold limit $\mathcal{F}$ diverges, and $\lambda_1 \sim |Z|/|\mathcal{F}|$, $\lambda_2 \sim |\mathcal{F}||Z|$. The measure for Z in this limit is $d|Z|^2 = |\mathcal{F}|^2 d\lambda_1^2$.

[5]However, the remaining terms may still fail to be positive.

As in the general discussion, we consider $|W|, |F| \ll 1$. Here this requires taking $|\mathcal{F}| \gg 1$. The constraint in eq. (2.11) forces $|Z|^2 \sim L$, so we write $Z = \sqrt{L}e^{i\phi}$. Then, at a metastable pure F-breaking critical point of V, $|F|^2/3 \sim |W|^2 = \lambda_1^2/4 \sim L/4|\mathcal{F}|^2$, which is indeed small when $\mathcal{F} \to \infty$. Granting $m_\pm^2 \sim F^2$ and substituting into eq. (3.8), we find

$$\begin{aligned} d\mu[F] &= \int d^2W\, d^2Z\, dF\, d\theta\, \delta(|Z|^2 - L)\delta(|F| - \sqrt{3}|W|)\delta(\lambda_1^2 - 4|W|^2) \times \\ &\quad \times \Theta_+(\theta, F)\frac{|Z|^4}{4|W|^2} \cdot \frac{(m_1^+)^2(m_1^-)^2}{2|F|^2} \\ &\sim \frac{3}{8}L^2\, d\phi\, d\theta\, \Theta_+(\theta, F) \times dF\, \delta\left(F - \frac{\sqrt{3}}{2|\mathcal{F}|}\sqrt{L}\right). \end{aligned}$$

Granting the behavior $\Theta_+ \sim |F|^2$, this looks like a scaling F^2dF at a given point in moduli space. What happened to the $|F|^5dF$ we found earlier? One factor $|F|^2$ was cancelled by the $|W|^2$ in the denominator — the general expectation that $|\det Z|^2$ would go as $|W|^2$ is violated in this case, because the see-saw mechanism pairs a large eigenvalue with the small eigenvalue. The remaining missing factor of F comes from the fact that in this particular ensemble, there is no additional suppression of small λ from the Z-measure, as Z does need to be tuned small to make λ small.

Finally, we need to incorporate the dependence of $\mathcal{F}$ on moduli. Suppose we are near a conifold limit, parameterized by a single complex structure modulus $t \to 0$; then

$$|F|^2 \sim |W|^2 \sim \lambda^2 \sim L|\mathcal{F}|^{-2} \sim L|t|^2 \log^3 |t|^2 \tag{3.14}$$

and the measure on moduli space $d^2z\sqrt{\det g}$ becomes

$$d^2t\ \log|t|^2 \sim \frac{d^2F}{L\log^2|F|^2}$$

leading to

$$d\mu[F] \sim L\frac{d^2F}{\log^2|F|^2}\Theta_+\,.$$

The final distribution (with the metastability constraint leading to $\Theta_+ \sim F^2$) goes as F^3dF. The difference with our previous general claim F^5dF arises from the see-saw factor $1/|F|^2$ in $|\det Z|^2$ (required by the scaling $\det Z \sim F^0$).

Thus, despite the fact that flux vacua are dual to gauge theory in the conifold limit, and do lead to hierarchically small scales, the final vacuum distribution does not show a corresponding enhancement at small scales. This came from a combination of effects, which we summarize again here and in the conclusions.

The hierarchically small scale is $|t|$, and it is true (eq. (3.14)) that this leads to $W \sim t$, which after imposing $\Lambda = 0$ implies $F \sim t$. On the other hand, the enhancement of the number of vacua near the conifold point found in [4] is not present for the small W component, in present terms because the large matrix element $\mathcal{F}$ cancels out of $\det Z$. In addition, the metastability condition eq. (2.28) gives an extra F^2.

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3.1.2 A simple but nongeneric multiparameter example: $T^6/\mathbb{Z}_2^2$

Another example which can be treated exactly is the orbifold $(T^2)^3/\mathbb{Z}_2^2$ (without discrete torsion). This has three complex structure moduli τ_i, and the only nonzero $\mathcal{F}_{IJK}$ is $\mathcal{F}_{123} = 1$. As usual we ignore the Kähler moduli. The Z-matrix is

$$Z = \begin{pmatrix} 0 & Z_1 & Z_2 & Z_3 \\ Z_1 & 0 & \bar{Z}_3 & \bar{Z}_2 \\ Z_2 & \bar{Z}_3 & 0 & \bar{Z}_1 \\ Z_3 & \bar{Z}_2 & \bar{Z}_1 & 0 \end{pmatrix}.$$

Its eigenvalues and eigenvectors can be obtained by elementary means. The eigenvalues are

$$\lambda_1 = |Z_1| + |Z_2| - |Z_3|\,, \qquad \lambda_2 = |Z_1| - |Z_2| + |Z_3|\,,$$
$$\lambda_3 = -|Z_1| + |Z_2| + |Z_3|\,, \qquad \lambda_4 = |Z_1| + |Z_2| + |Z_3|\,,$$

and the eigenvectors ψ_A have simple expressions which depend only on the phases ϕ_i of Z_i:

$$(\psi_A)^B = \frac{i}{2}\begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ i & i & i & i \end{pmatrix}\begin{pmatrix} e^{i(\phi_1+\phi_2+\phi_3)/2} & 0 & 0 & 0 \\ 0 & e^{i(\phi_1-\phi_2-\phi_3)/2} & 0 & 0 \\ 0 & 0 & e^{i(-\phi_1+\phi_2-\phi_3)/2} & 0 \\ 0 & 0 & 0 & e^{i(-\phi_1-\phi_2+\phi_3)/2} \end{pmatrix}.$$

Note that since the Z-measure is just d^6Z, there is no suppression of coincident or zero eigenvalues. This can be understood along the lines of the discussion at the end of 2.4, by noting that the matrices $\hat{Z}_i = \partial Z/\partial|Z_i|$ satisfy eq. (2.33) with $U_A^B = (\psi_A)^B$ independent of $|Z_i|$, and thus can be simultaneously diagonalized on a three real dimensional slice of parameter space.

This feature will be destroyed by generic perturbations of the prepotential. A simple example for which this is easily verified is obtained by adding "conifold-like" terms $\mathcal{F}_{IJK} = \delta_{IJ}\delta_{IK}/t_K$. Thus, we believe the present example is not typical, and that more generic flux ensembles will share the CI ensemble behavior. However, let us finish the discussion for completeness.

Putting $F = \psi_1$, eq. (3.13) gives

$$(m_1^+)^2 = 2|F|^2\,, \qquad (m_1^-)^2 = 4(|W|^2 - |F|^2)\,.$$

This simplification and in particular independence of θ and vanishing of the $O(F)$ term occurs because $D\mathcal{F} = 0$ in this case. This is another nongeneric feature of this model. One consequence is that there is less tuning control of the masses, and in particular in the case of pure F-breaking and approximately vanishing cosmological constant, $(m_1^-)^2 = -8|F|^2/3 < 0$, so there are no such metastable vacua. When a constant D-term is added, this becomes $(m_1^-)^2 = 4(D^2 - 2|F|^2)/3$, which is positive for sufficiently small F.

Plugging this in the general formulae, we find for the vacuum density at zero cosmological constant

$$d\mu \sim d\Lambda\, d^2W\, d^6Z\, dF\, \delta(|F|^2 + D^2 - 3|W|^2)\, \delta(\lambda_1^2 - 4|W|^2) \times$$

$$
\begin{aligned}
&\times\Theta(|W|-|F|)\cdot(\lambda_2\lambda_3\lambda_4)^2\cdot\frac{(m_1^+)^2(m_1^-)^2}{|F|}\\
&\sim d\Lambda\, dF\,\frac{|F|(D^2-2|F|^2)}{\sqrt{|F|^2+D^2}}L_*^{11/2}\,.
\end{aligned}
$$

The exponent of L_* can be checked by counting dimensions, where F, W, Z get assigned dimension 1, and thus L, Λ get dimension 2. The total dimension must be $2b_3 = 16$, which is indeed the case with the above power of L_*.

When $F \ll D \sim W$, this goes as dFF, which is as in the generic case but with an extra factor of F, due to the fact that $(m_1^+)^2$ is of order F^2 here. When $F \sim D$ (but not too close to $D/\sqrt{2}$), this goes as dFF^2. This has two powers less than the generic case, one because there is no additional suppression here of small λ, and one because no fine-tuning is needed to kill the $\pm O(F)$ term in the masses. Finally, when $|F| \to D/\sqrt{2}$, the density drops to zero together with the mass m_1^-.

Finally, there would also be terms from the branches with degenerate eigenvalues, of the same form multiplied by powers of $F/\sqrt{L}$. If we trust the results all the way to $|F| \sim \sqrt{L}M_{string}^2$, which might be reasonable for $M_{string} \ll M_p$ (as at very weak string coupling), these will be comparable in number to the vacua we just described; however as we discussed their existence appears to be a special feature of this example.

3.1.3 The generic multiparameter case

We now discuss what we would expect to see for a more generic multiparameter case, in the continuous flux approximation. To get low scale supersymmetry breaking with zero c.c., we need small W amd F. While in this approximation, there is no *a priori* constraint on the size of these parameters, solving the equations eq. (2.13) requires the matrix Z to have a small eigenvalue.

It is hard to tune a generic matrix to get a small eigenvalue. This is easily made precise in the standard ensembles of random matrices. In the CI ensemble, the measure for matrices with a small eigenvalue λ goes as $d(\lambda^2)$. Following this through leads to the generic $F^5 dF$ distribution we discussed above.

In the one parameter case, Z is a 2×2 matrix, and the dilaton dependence of the IIb flux superpotential leads to a see-saw structure. This is special to 2×2 matrices. More generally, what we can accomplish by tuning to conifold points is to get large matrix elements in Z, for example the $\mathcal{F}Z \sim Z/t$. In higher dimensions, large matrix elements do not generally lead to small eigenvalues.

A simple model which may illustrate the general situation is to take Z to have generic order 1 coefficients except for a single large matrix element $Z_{11} = \mathcal{F} \gg 1$. The eigenvalues of such a matrix can be found by treating the $O(1)$ coefficients as a perturbation around the spectrum of the matrix for which only $Z_{11} = \mathcal{F}$ is non-zero. A generic $O(1)$ perturbation will shift these eigenvalues by $O(1)$, and the resulting eigenvalues will be $\mathcal{F} + O(1)$ and the rest $O(1)$.

The see-saw matrix escapes this general result because $Z_{00} = 0$ forces $\det Z \sim 1$ and thus a small eigenvalue must appear. To get this effect in a higher dimensional matrix, the determinant of the $1,1$ minor of Z must vanish. This would appear to be a rather complicated and non-generic tuning.

Thus, we see no clear way out of the generic $F^5 dF$ prediction of our earlier discussion in this case. Since even in the one parameter case, the expected enhancement of low scale vacua was cancelled by measure factors, the suppression of low scale vacua of this type would appear quite general.

3.2 Quantized fluxes

As in [2, 4], we have ignored flux quantization in the results so far, instead taking the fluxes to be continuous variables. At first sight this may seem a drastic simplification as the allowed values of the EFT parameters (W, F, Z) are heavily influenced by flux quantization. For example, the hierarchically small value of certain contributions to the superpotential near the conifold point is only apparent in the quantized flux problem.

Now, as in [2, 4], one can argue on general grounds that the approximation of taking flux continuous reproduces the leading large L asymptotics for the number of vacua, because the volume of the region in flux space supporting vacua is the asymptotic for the number of quantized flux vacua (lattice points contained in the region). We make this argument below. Furthermore, explicit numerical study [25, 4, 6, 26] has confirmed the validity of the approximation in counting supersymmetric vacua when $L \geq K/r^2$, to estimate the number of vacua in a region of moduli space of radius r, in finding the distribution of W, and other observables.

This does not mean that we should immediately accept the analogous claims for non-supersymmetric vacua, as there are clearly important differences between the two problems. For example, the validity of statistical approximations to obtain the W distribution is surely helped by the fact that W is a sum of complex quantities with arbitrary phases, which typically involves many cancellations. On the other hand, since the quantity M^4_{susy} is a sum of squares, errors in the approximation might lead to systematic overestimates.

More work will be needed to find the minimal flux L for which the results here are accurate, but let us proceed to make some general comments.

3.2.1 Geometry in flux space

We now describe the region in type IIb flux space which supports nonsupersymmetric vacua, in the same sense developed for supersymmetric vacua in [2].

We work around a point z in moduli space and make the appropriate decomposition (W, F, Z) of the fluxes at that point. We then think of the (W, F) directions as fibered over the Z plane. At a given Z and θ, the branches of nonsupersymmetric vacua sit at the m values of $|W|$ given by the eigenvalues of Z, and in one-dimensional subspaces of the F space given by the eigenvectors. These m solutions of the eigenvalue equation or sheets are the basic "branches of solutions" in this problem. Of these, only the one with the lowest value of $|W|$ is tachyon-free. Moving around in the Z plane varies them, and one can have monodromies about points with degenerate eigenvalues which exchange sheets.

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Varying θ then fills out a correlated circle in W and F, to give two real dimensional sheets in the $2m+2$ dimensional (W, F) space. Finally, varying the $2m$ moduli will turn these sheets into cones or balls of full dimension.

Another way to think about this is to consider the "universal" solution space as (W, F) fibered over the entire $m(m+1)$ real dimensional space of Z, which contains a real codimension $2m$ domain of allowed nonsupersymmetric vacua (say with $|F|^2$ and $|W|^2$ bounded) which looks like a two dimensional sheet in (W, F). Each point in moduli space then produces a rank $4m$ lattice (in the IIb theory; on fourfolds it could be larger) in this space, and then varying the $2m$ real moduli allows these to hit the universal solution space at isolated points, the non-supersymmetric vacua.

In any case, the region in flux space which supports nonsupersymmetric vacua is of full dimension in the flux space and has a smooth boundary, and thus standard arguments imply that the leading large L asymptotic for the number of lattice points contained in this region will be its volume.

3.2.2 Subleading components in L

Perhaps a subleading term at large L has a different F distribution, for example producing many more small F vacua, in a way which makes it dominate in the physical regime.

While the true distribution of F is quantized, one would expect this to cut out small F vacua. In particular, one might naively expect the distribution at a value F to be well approximated by the large L asymptotic only when $L > 1/|F|^2$. On the other hand, small parameters enter into this relation, as we saw in the explicit one parameter example, so the situation in cases of interest is probably better than this. In any case, this effect will shift the F distribution towards higher scales.

On the other hand, it has been suggested by Dine *et al* [19] that the W distribution could obtain a component highly peaked at zero, which after tuning the c.c. would lead to a peak in F. Their idea is that, because $W = 0$ restores R symmetry, there might be an enhanced number of flux vacua with $W = 0$. Then, since our exact considerations are of course just approximations to the full physical problem, one might expect such a peak to be smoothed out to an enhancement of small $|W|$ vacua in the full theory.

Recently DeWolfe *et al* [26] have studied the problem of finding supersymmetric flux vacua with $W = 0$ in some detail and indeed find enhanced numbers of $W = 0$ vacua at subleading order in L in simple examples. While it remains to be seen how important this effect is, the idea and evidence for it seem quite reasonable. However, it seems unlikely that this comes with an enhancement of small W vacua in the pure flux vacuum problem. This is for both mathematical reasons (the points with an enhanced number of vacua tend to be surrounded by "voids" without vacua), and physical reasons — while the flux vacua do contain exponentially small effects, the superpotentials which stabilize all moduli also contain other $O(1)$ contributions which show up in W and cannot be eliminated.

Rather, one needs to call on additional physical effects to smooth out the distribution. At first sight the suggestion of [19] seems plausible: exponentially small corrections could lead to vacua with small $W \sim \exp -1/g^2$, and a uniform distribution of couplings dg would translate into a logarithmic distribution dW/W. However, at present we know of no explicit ensemble of models in which this idea would be realized. In particular, there is no evidence that the KKLT construction [20] would do this. While it relies on exponentially small effects to stabilize Kähler moduli, these are balanced against a preexisting small W obtained from the flux sector, and the construction does not work without this. One might hope that a model more like the original "racetrack," in which several competing exponentials stabilize the Kähler moduli, could lead to small W, but if the potential is naturally a polynomial in the exponentials $q_i \equiv \exp -1/g_i^2$, one might expect the resulting distributions to be uniformly distributed in the variables q_i. All this is not to say the suggestion is clearly false, but rather that to support it one needs to show that it is realized in some explicit ensemble of models which could plausibly come out of string compactification.

In our opinion, at present the best motivated conjecture one could make for such an ensemble of Kähler stabilized models, is simply that it is similar to one of the known flux ensembles. As in [1], one might try to argue this from the existence of gauge-flux dualities such as [27] which (in much simpler examples) explicitly relate the two classes of vacua. Not having a compelling argument of this type, we will simply make the comment that if the true ensemble of Kähler stabilized IIb models turns out to be different, one will also want to understand why this type of duality argument fails.

Suppose we were to grant that there is a large component of vacua with small W distributed as $d^2W/|W|^2$; how would this influence our results? We need to first ask if this component is already visible in our computations, and properly taken into account. Although there are similar looking factors in our intermediate steps, in fact they are not present in the final results, for the reasons we explained. Rather, we would interpret the suggestion of [19] as saying that considerations in a sector of the theory not considered here lead to an additional factor $|W|^{-2}$ in the original distribution eq. (2.11). If we incorporate such a factor, given $F \sim W$ we will find the generic distribution changes from $F^5 dF$ to $F^3 dF$. This still appears to favor the high scale, at least for the purely F breaking vacua.

4. Including D-terms

At present there is no explicit ensemble of EFT's with D terms which looks simple and universal enough to justify a detailed study. Thus in this section we simply add a generic additional contribution to the potential,

$$V = |F|^2 - 3|W|^2 + \epsilon V_D \, ,$$

and discuss its consequences.

To get small supersymmetry breaking, we take ϵ to be small. This would be the case for instance for a contribution arising from an anti-D3 brane placed at the bottom of the warped throat developing near a conifold degeneration [11]. In this case, which will be our concrete example below, $\epsilon \sim |v|^{4/3}$, where v is the complex structure coordinate given by the period of the vanishing 3-cycle.[6]

The condition for a critical point $\nabla V = 0$ becomes

$$N \begin{pmatrix} e^{-i\theta} F \\ e^{i\theta} \bar{F} \end{pmatrix} = -\nabla(\epsilon V^D) \equiv -\epsilon d$$

with N as in eq. (2.13). This is solved by

$$\begin{pmatrix} e^{-i\theta} F \\ e^{i\theta} \bar{F} \end{pmatrix} = -\epsilon N^{-1} d \,. \tag{4.1}$$

Decomposing $d = d_A^+ \Psi_A^+ + d_A^- \Psi_A^-$ with $d_A^\pm \in \mathbb{R}$ and $\Psi_A^\pm$ as defined in eq. (2.16), we have

$$|F|^2 = \epsilon^2 \sum_A \frac{(d_A^+)^2}{(\lambda_A - 2|W|)^2} + \frac{(d_A^-)^2}{(\lambda_A + 2|W|)^2} \,. \tag{4.2}$$

4.1 Lifted susy vacua

Generically (i.e. for λ_A not too close to $2|W|$), we can assume all terms in the sum to be at most of order 1. This assumption needs some discussion for the antibrane example because of the dependence of ϵ on the modulus v, which implies $d_v \sim 1/v$. However, the matrix element $Z_{vv} \approx \mathcal{F}_{vvv} \bar{Z}^v \sim 1/v$ in N compensates for this. In other words, there will be a pair of eigenvectors $\Psi_v^\pm$ approximately associated to the v-direction, and although in this direction $d_v \sim 1/v$, we also have $\lambda_v \sim 1/v$, and the two cancel against each other in eq. (4.2).[7] Thus, $|F|$ is generically of order ϵ.

In this case, the D-term dominates over the $|F|^2$-term in the potential:

$$V \approx -3|W|^2 + \epsilon V_D \,,$$

and the measure becomes

$$\begin{aligned} \delta^{2m}(dV) |\det V''| &= \frac{\delta^{2m}(F_A + \epsilon\, e^{i\theta} (N^{-1} d)_A)}{|\det N|} \,|\det HN(\mathbf{1} + (HN)^{-1}(V_1'' + V_2'' + \epsilon V_D''))| \\ &\approx \delta^{2m}(F_A + \epsilon\, e^{i\theta}(N^{-1}d)_A) \,|\det H| \,. \end{aligned} \tag{4.3}$$

Again some discussion is needed to justify dropping the V_D'' term, since it blows up as $1/v^2$ in the v-direction. However in this direction $HN \sim \lambda_v^2 \sim 1/v^2$ as well, so this cancels out.

The above expression is exactly the density for the supersymmetric branch. Similarly, the condition for the absence of negative modes of V'' is the same as for the superymmetric case: $\lambda_A > 2|W|$, up to possible corrections from V_D'', but if $\lambda_A - 2|W| \gg \epsilon$ these corrections will be small.

---

[6] While it has been argued [28, 29] that if the anti-D3 contribution has an EFT description, this must be a D term, we know of no description within the usual rules of $\mathcal{N} = 1$ supergravity which satisfies all known properties of the string theory construction. However this point will not be crucial for what follows.

[7] We should actually go to an orthonormal frame first, but since the metric $ds^2 \sim \log|v|^{-2} dv d\bar{v}$ in the v direction, this only induces logarithmic corrections.

Thus, these nonsupersymmetric vacua correspond to supersymmetric vacua "lifted" by the D-term. The number density of such vacua at cosmological constant Λ will approximately be equal to the number density of supersymmetric vacua at susy cosmological constant $\Lambda - \epsilon V_D$. Since the latter is nonvanishing at zero [4], we have that for small $\Lambda < \epsilon V_D$, this density is essentially independent of Λ, and roughly equal to $1/\mathcal{N}_{\rm tot}(\mathcal{R})$ when integrated over a region $\mathcal{R}$ in moduli space.

The supersymmetry breaking scale $M_{\rm susy}$ for these vacua will be of order $\sqrt{\epsilon}$, and thus the expected number of these "lifted susy" vacua with $M_{\rm susy} < M_*$ is, using the results of [4] for the susy vacuum distribution near the conifold,

$$\mathcal{N}(M_{\rm susy} < M_*) \sim \frac{1}{\log M_*}\,. \tag{4.4}$$

That is, taking into account tuning of the Higgs mass, for this family of near-conifold lifted susy vacua, a low susy breaking scale is favored.

4.2 Perturbed F-term susy breaking vacua

The situation changes when one of the eigenvalues λ_A approaches $2|W|$. Since we turned on the D-term potential as a small perturbation, we still expect that $2|W|$ will be the generic approximate lower bound on the λ_A to get a metastable minimum. Let us therefore assume that all λ_A are well above $2|W|$ except possibly λ_1. Define $u \equiv \lambda_1 - 2|W|$. When $u \ll 1$, we have

$$F \sim \frac{\epsilon\, d_1^+}{u} \sim \frac{\epsilon}{u}\,. \tag{4.5}$$

Note that since we are interested in $F \ll 1$, we require $u \gg \epsilon$. All $(m_A^\pm)^2$ for $A > 1$ as defined in eq. (2.22) will still be given approximately by their susy values. On the other hand

$$(m_1^+)^2 = (u + 3|W|)u + c_1\frac{\epsilon}{u} + c_2\frac{\epsilon^2}{u^2} + c_3^+\epsilon \tag{4.6}$$

$$(m_1^-)^2 = (u + |W|)(u + 4|W|) - c_1\frac{\epsilon}{u} + c_2\frac{\epsilon^2}{u^2} + c_3^-\epsilon\,, \tag{4.7}$$

where the c_i are generically of order 1. We consider three regimes:

1. $\epsilon^{1/3} \ll u \ll 1$.

   In this case, the first term in eq. (4.6) is much bigger than $\epsilon^{2/3}$, and the remaining terms are all much less than $\epsilon^{2/3}$. The same is true for eq. (4.7). Therefore we can drop all correction terms and we are back in the previous case of D-term lifted supersymmetric vacua.

2. $\epsilon \ll u \ll \epsilon^{1/2}$.

   When we don't impose constraints on the cosmological constant, so we can assume $|W|$ to be generic, we have that the first term in eq. (4.6) is much smaller than $\epsilon^{1/2}$, while $c_1\epsilon/u \gg \epsilon^{1/2}$. The other terms are again much smaller, so m_1^+ is dominated by the second term. Furthermore $(m_1^-)^2 \approx 4|W|^2$. This is as in the case of pure F-term susy breaking.

The case with cosmological constant constrained near zero is similar. Since $\epsilon^2/u^2 \gg \epsilon$, the $|F|^2$-part of the potential will dominate over the D-term part, and thus to get near zero vacuum energy we should take $|W| \sim F \sim \epsilon/u \gg u$. This implies that the first term in eq. (4.6) is of order ϵ and can be dropped together with the D-term, bringing us again to the situation of pure F-breaking. For m_1^- similar considerations hold.

Hence for either case the measure $\delta^{2m}(dV)|\det V''|$ becomes

$$\delta^{2m}(F+\epsilon N^{-1}d)\prod_{A>1}(\lambda_A^2-|W|^2)\cdot\frac{(m_1^+)^2(m_1^-)^2}{u(u+4|W|)}\approx$$
$$\approx \delta^{2m}(F+\epsilon N^{-1}d)\frac{|\det H|}{3|W|^2}\cdot\frac{(m_1^+)^2(m_1^-)^2}{4|W|u} \tag{4.8}$$

with $m_1^{\pm}$ approximated as in the pure F-breaking case. Note furthermore that the delta-function forces F to lie approximately in the direction of the eigenvector Ψ_1^+ as $u = \lambda_1 - 2|W|$ is small. Finally, since $F \sim \epsilon/u$, we can write

$$\frac{d\lambda_1}{u}=\frac{du}{u}=\frac{dF}{F}$$

and change variables from λ_1 to F (mapping the domain $2|W|+\epsilon \ll \lambda_1 \ll 2|W|+\epsilon^{1/2}$ to $\epsilon^{1/2} \ll F \ll 1$). This completely reproduces the measure eq. (3.4) of pure F-breaking vacua in this regime. Thus, pure F-breaking vacua with susy breaking scale $M_{\rm susy} = F \gg \epsilon^{1/2}$ are just slightly perturbed by adding a D-term potential ϵV_D, and their distributions remain essentially identical.

Note that because of the suppression by at least F_*, these vacua can be expected to be significantly less numerous than the lifted supersymmetric ones.

3. $\epsilon^{1/2} < u < \epsilon^{1/3}$.

   This is the intermediate regime, where D-term and F-term effects are of comparable size. When ϵ is small, this corresponds to only a small fraction of susy breaking vacua, compared to the other two regimes.

4.3 Summary

Adding a D-term which becomes small in a region $\mathcal{R}$ of moduli space will remove all vacua located outside that region from the ensemble of vacua with "small" susy breaking scale ($M_{\rm susy} \ll m_p$), simply because the D-term either destabilizes the vacuum, or it causes the susy breaking scale to be too large. Within $\mathcal{R}$, it will add new susy breaking vacua obtained by lifting susy vacua.

For the particular case of susy breaking by an anti D3-brane near a conifold point, the number of vacua with susy breaking scale $M_{\rm susy} < M_*$ will approximately be given by

$$\mathcal{N}_0(M_{\rm susy} < M_*) \sim \frac{\mathcal{N}_{susy}(\mathcal{R})}{\log M_*} \tag{4.9}$$

where $\mathcal{N}_{susy}(\mathcal{R})$ is the number of tachyon-free susy vacua in $\mathcal{R}$. Restricting the cosmological constant to a small interval of width $\Delta\Lambda$ around zero just multiplies this number by $\Delta\Lambda$. Apart from these lifted susy vacua, descendants of pure F-term breaking vacua are also present and are only slightly perturbed as long as $|F|^2$ is bigger than the added D-term potential. Their number can be expected to be roughly

$$\mathcal{N}_F(M_{\rm susy} < M_*) \sim M_*^2\, \mathcal{N}_{susy}(\mathcal{R}) \tag{4.10}$$

without constraint on Λ, and

$$\mathcal{N}'_F(M_{\rm susy} < M_*) \sim M_*^{12}\, \Delta\Lambda\, \mathcal{N}_{susy}(\mathcal{R}) \tag{4.11}$$

if Λ is constrained near zero.

Finally, although detailed considerations in the observable sector are beyond our scope here, one effect which must be mentioned is the universal $-2|W|^2$ contribution to the bosonic mass matrix eq. (2.4), which will destabilize vacua with light fermions and $|D|, |W| \gg |F|$, thus removing a large class of potentially realistic vacua.

5. Conclusions

We derived formulae for the distribution of nonsupersymmetric vacua in a general ensemble of effective supergravity theories, such as eq. (3.4). This distribution is rather similar to that for supersymmetric vacua, with certain "correction factors" which we explained.

We began by reformulating the problem of finding nonsupersymmetric F breaking vacua as that of finding eigenvectors of the matrix D^2W of second derivatives of the superpotential. This makes several features of the problem manifest, most importantly that metastable nonsupersymmetric vacua are generic, of number comparable to the number of supersymmetric vacua.

We then argued that the suggestion of [4, 17] for a large power law growth in the number of F breaking vacua in models with many moduli is not expected in general. While the heuristic argument leading to this suggestion seems sensible, namely that independent supersymmetry breakings in independent hidden sectors combine additively and favor high scale breaking, our detailed analysis shows that the different F terms will be independently distributed only when the different hidden sectors are totally decoupled, which seems unlikely to us based on our studies so far. It should be said that this does not address the similar argument in [18] in terms of susy breaking by multiple antibranes, and the related idea in [17] that multiple independent D terms could lead to the same effect, as the details there are quite different, and this possibility remains open.

We went on to compute the density of nonsupersymmetric vacua, eq. (3.1), in general and in simple examples. Much of the structure of the result comes from the factor $\det V''$ which is included to normalize the delta function $\delta(V')$ to one for each vacuum. Physically, this factor is the product of masses squared for all bosons, and this makes precise the general expectation that bosons much lighter than the natural scale of the potential (for flux potentials, the string scale) are disfavored.

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Perhaps the simplest summary of the results for pure F breaking is eq. (3.9), which states that the number $\mathcal{N}$ of pure F breaking vacua with supersymmetry breaking scale $M_{\rm susy} \leq M_* \ll M_{pl}$, and with $\Lambda \sim 0$, goes as $\mathcal{N} \sim M_*^{12}$. This somewhat surprising claim is made up of a number of factors, explained in detail in section 3:

- The most naive expectation would have been a uniform distribution d^2F for the complex parameter F, leading to $\mathcal{N} \sim M_*^4$. However, as discussed in section 2.2, the equations $V' = 0$ determine the phase of F, leading instead to the generic distribution dF and the counting $\mathcal{N} \sim M_*^2$, for vacua in which the c.c. is not tuned, or in which it is tuned by other effects (say by D terms).
- If we tune the c.c. to zero using $|F|^2 = 3|W|^2$, we need to take the W distribution into account. W is a complex variable, and is proportional to the lowest eigenvalue λ of the matrix $Z \equiv D^2W$ (in the sense of eq. (2.17)). Since Z is complex symmetric, this is distributed as $d(\lambda^2)$, leading to an extra factor of $\lambda \sim W \sim F \sim M_*^2$.
- The measure factor $\det V'' = \prod_i m_i^2$ weighs the density of vacua with the product of the masses squared of every boson. As discussed in section 2.3, the two bosonic partners to the goldstino generically have masses determined by W and F, and this leads to a factor $F^2 \sim M_*^4$.
- Finally, metastability requires a tuning which leads to an additional factor $F^2 \sim M_*^4$, as discussed in section 2.3.

In sum, enforcing $\Lambda \sim 0$ has a significant effect on the distribution, apparently favoring high scale breaking enough to outweigh factors such as eq. (1.2) and the suggestions discussed in section 3.2.2. Note that the arguments only really require $\Lambda < |F|, |W|$, which is good as any more precise requirement would be spoiled by subsequent corrections.

We can compare this to a rough picture of the "D breaking" vacua. Since the D parameters are real, and this type of supersymmetry breaking need not come with light scalars, one does not get the measure factors we just discussed, leading to distributions such as eq. (4.4) which would seem to favor low scale breaking. On the other hand, multiple D parameters might still lead to power law growth in $M_{\rm susy}$, and a full analysis might suggest other measure factors, so this remains unresolved. We also do not know whether F or D breaking is more common, since we have as yet no explicit ensemble which includes both types of vacuum.

All of these considerations are of course within a subsector of the theory and ignore many further corrections. Nevertheless the factors we just listed which entered in our final results appear fairly generic, and thus we believe they will be important ingredients in all computations of this type. Numerically, they appear as important as the specific physical effects focused on in previous work.

Much more detailed work in examples will be required to judge to what extent the full picture is controlled by such generic features. One can take various attitudes about them — perhaps they are background, perhaps they are the signal. But they appear to us to be very basic properties of the distribution of string/M theory vacua.

Of course, we are still only talking about a small part of the full problem of counting realistic string/M theory vacua. One might expect equally important selection factors to arise at subsequent stages, and it is probably premature to read too much phenomenological significance into our results so far. But we would like to suggest that analyses at this level of detail, at least for the best understood classes of vacua (Calabi-Yau compactification of IIb, F and heterotic theories), for every part of the full problem, might be feasible over the next few years, and that this would enable us to make similar statements with some claim to significance.

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